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Statistical-mechanical formulation of Lyapunov exponents

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Abstract

We show how the Lyapunov exponents of a dynamic system can, in general, be expressed in terms of the free energy of a (non-Hermitian) quantum many-body problem. This puts their study as a problem of statistical mechanics, whose intuitive concepts and techniques of approximation can hence be borrowed.

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1. Introduction

Lyapunov exponents are an important tool for the characterization of dynamical systems. Their very definition has a strong statistical-mechanical flavour, as it involves ‘extensivity in time’ of certain quantities regardless of the initial (‘border’) conditions. It is then natural to seek to express them in an explicitly statistical-mechanical way, in terms of a partition function.

In this paper, building upon previous work, mainly of Graham [1] and Gozzi and Reuter [2], we show how the study of Lyapunov exponents can be cast as a quantum many-body problem—in fact a rather standard one, except that the Hamiltonian is in general non-Hermitian. As we shall see, this does not bring a miraculous solution to all calculational problems, but it serves two purposes:

- Because the problem is formulated as a standard quantum many-body one, all the tools developed in that wider context are available. Some, but not all, of them have already been used as approximation schemes for Lyapunov exponents, but others such as Hartree–Fock, dynamic mean-field theory and the renormalization group seem promising.
- More important, general theoretical results are re-expressed in a language that is often familiar. A typical example is when one asks whether finite-dimensional systems have a limit Lyapunov density function $\rho(\lambda)$: in this setting the question becomes whether a quantum finite-dimensional system has an extensive free energy for all chemical potentials. Although this does not in itself prove the existence of $\rho(\lambda)$ in the thermodynamic limit, it renders it intuitive and acceptable, at least up to the level of rigour of theoretical physics.

Consider a general dynamical system,

$$\dot{x}_i = f_i(\mathbf{x}, \eta) \quad i = 1, \dots, N \quad (1)$$

and, in particular, the version with additive noise

$$\dot{x}_i = f_i(\mathbf{x}) + \eta_i. \quad (2)$$

where η_i is a Gaussian white variable with variance $2T$. In the limit of zero noise we have a standard dynamical system and for a particular form of f a Hamiltonian system. Clearly, an alternative way to study (2) is to go to the Fokker–Planck (or ‘Kramers’, or ‘Liouville’, depending on the context) description of the evolution of the probabilities

$$\dot{P}(\mathbf{x}, t) = -H_{\text{FP}}P(\mathbf{x}, t) \quad (3)$$

where

$$H_{\text{FP}} \equiv -\frac{\partial}{\partial x_i} \left(T \frac{\partial}{\partial x_i} - f_i \right). \quad (4)$$

Here and in what follows summation of repeated indices is assumed, unless otherwise stated. The Fokker–Planck operator acts on the space of functions of the coordinates \mathbf{x} : it resembles a Schrödinger operator, although it is in general non-Hermitian. The noiseless limit is subtle, and is the subject of ergodic theory.

Introduce now two sets of fermion a_i^\dagger, b_i^\dagger and boson $\alpha_i^\dagger, \beta_i^\dagger$ creation operators ($i = 1, \dots, N$); and the corresponding vacuum, defined by

$$a_i|-\rangle = b_i|-\rangle = \alpha_i|-\rangle = \beta_i|-\rangle = 0 \quad \forall i. \quad (5)$$

It will turn out that all the information we search for is obtained directly from the following generalization of the Fokker–Planck operator,

$$H_S = H_{\text{FP}} - V_{kl}(\mathbf{x})(a_k^\dagger a_l + b_k^\dagger b_l + \alpha_k^\dagger \alpha_l + \beta_k^\dagger \beta_l) \quad V_{kl}(\mathbf{x}) \equiv \frac{\partial f_k}{\partial x_l}. \quad (6)$$

Here H_S acts on the product space of functions of the coordinates and of the number of fermions and bosons of each type. Clearly, H_S coincides with H_{FP} when restricted to the zero fermion and boson subspace. *This is the many-body system mentioned above.*

If the x_i are lattice variables, and the f_i are short-range interactions, then the system (6) defines a quantum (non-Hermitian) theory, also having short-range interactions. If the system is instead off-lattice, the x_i describe the positions of the particles and the $(a_i, b_i, \alpha_i, \beta_i)$ play the role of ‘spin’ degrees of freedom carried by the quantum particles: if the f_i are short range, then both the direct and the spin–spin interaction are also short range.

An alternative strategy, that leads to the generalized Lyapunov exponents, is based on replicas of fermions $a_i^{\gamma\dagger}, b_i^{\gamma\dagger}$ with $\gamma = 1, \dots, q$ and the operator

$$H_q = H_{\text{FP}} - V_{kl}(\mathbf{x})(a_k^{\gamma\dagger} a_l^\gamma + b_k^{\gamma\dagger} b_l^\gamma). \quad (7)$$

The Lyapunov exponents are defined as follows: the separation of two infinitesimally close trajectories evolving *under the same noise* (for a mathematical review see [3]) $\mathbf{x}(\mathbf{t})$ and $\mathbf{x}(\mathbf{t}) + \mathbf{y}(\mathbf{t})$ is described by the evolution of vectors in the tangent space

$$y_k(t) = U_{ki}(\mathbf{x}_0, t)y_i(0) \quad (8)$$

where \mathbf{x}_0 is the initial condition and U is defined as the solution of the linear equation

$$\dot{U}_{ki}(t) = V_{kj}(\mathbf{x}, t)U_{ji}(t). \quad (9)$$

The N Lyapunov exponents λ_i measure the rate of growth of the volume in the tangent space. We construct

$$A(t) \equiv U(t)U^\dagger(t) \quad (10)$$

and consider its eigenvalues $\mathcal{A}_1(t) \geq \mathcal{A}_2(t) \cdots \geq \mathcal{A}_N(t)$ (a set per initial condition and/or noise realization). The Lyapunov exponents are

$$\lambda_i \equiv \lim_{t \rightarrow \infty} \frac{1}{2t} \langle \log \mathcal{A}_i \rangle. \quad (11)$$

Throughout this paper averages $\langle \bullet \rangle$ are over the noise and/or the initial condition. Note that the λ_i are automatically labelled in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Their existence has been proved for dynamical systems by Oseledec [4] and his work has been extended to a very large class of systems (see the introductory article of Arnold in [3] and the references therein). One can in fact consider the generalized Lyapunov exponents λ_i^q of interest as a measure of the intermittency (see Benzi *et al* [5]) defined, for example, as

$$\lambda_i^{2q} \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \log \langle (\mathcal{A}_i)^q \rangle \quad (12)$$

(a more standard definition will be given below in (17)).

Adopting the nomenclature of disordered systems, we shall call the true Lyapunov exponents ‘*quenched*’, and the $q = 1, 2, \dots$ set the ‘*annealed*’ exponents. The quenched exponents can be formally obtained as the analytic continuation $q \rightarrow 0$ (the replica trick), but we shall also consider a direct evaluation here.

Years ago, Graham [1] constructed a representation of the *annealed* Lyapunov exponents using fermions valid for systems with one degree of freedom. Later, Gozzi and Reuter [2] derived a generalization to several degrees of freedom. Their approach differs from ours in two respects: first, their expression, if averaged over initial conditions or noise, yields the annealed quantities. Second, and also important, they consider the eigenvalues of U rather than UU^\dagger , a different (though potentially interesting) quantity. In the present treatment, we derive an expression for the quenched exponents directly by introducing auxiliary fermions *and bosons*: the technique of supersymmetry [6, 7]. We also give expressions for the generalized exponents for even q . In all cases we consider the operator UU^\dagger , rather than U : this is the reason why we need to introduce the fermions and bosons in pairs (a_i, b_i) and (α_i, β_i) (cf (6), (7)).

The present derivation is confined to continuous-time systems; it would need to be modified in order to apply it to discrete maps. The presence of noise is an important ingredient of our formalism, in that it provides a well-defined Hilbert space structure. The zero-noise limit can be taken but brings in all the subtleties of ergodic theory.

Lyapunov exponents appear naturally in the context of localization problems for quantum disordered systems: one considers exponents associated with the product of transfer matrices, which play the role of evolution operators (see, e.g., [8, 9]). Thus, the Green-function formalism has been treated with supersymmetric (SUSY) techniques to extract correlations (see Balents and Fisher [10], where there is also a detailed analysis of the supersymmetry group and representation, and Markoš [11] for a path-integral formalism). In these works supersymmetry was used in order to obtain the Green function, and then the quenched quantities were obtained via the replica trick.

The observation that certain quantities in the theory of deterministic dynamical systems have a statistical-mechanical interpretation is also at the heart of the so-called ‘thermodynamical formalism’ [12]. The formalism we present here is not equivalent and is more along the lines of the approach in [13] where the problem of computing the Lyapunov exponents is mapped into a statistical-mechanical problem.

This paper is organized as follows. In section 2 we present the formalism for the expression of usual and generalized Lyapunov exponents. For Hamiltonian systems we show (in section 3) that the Lyapunov pairing rule follows trivially from a particle–hole symmetry. In section 4 we discuss the statistical mechanics of Lyapunov exponents of macroscopic systems, in particular

the existence of a Lyapunov density function. We then present two examples of application: in section 5 to problems of random matrices, and in section 6 to a Hamiltonian mean-field model.

2. The formalism

2.1. Basic quantities

The average expansion of any p -dimensional volume evolving with (2) in the space of phases can be expressed, in terms of U as

$$\det [(\xi_{i_1}^\dagger, \dots, \xi_{i_p}^\dagger)^\dagger [U^\dagger U](x_0, t) (\xi_{i_1}, \dots, \xi_{i_p})] \quad (13)$$

where $\xi_{i_1}, \dots, \xi_{i_p}$ is a set of p orthonormal vectors. Given that the spectrum of $U^\dagger U$ is generically the same as the spectrum of UU^\dagger the average expansion can be expressed as the sum over all possible p -dimensional volumes

$$R_p(\mathbf{x}_0, t) = \sum_{i_1, \dots, i_p} \det [(\xi_{i_1}^\dagger, \dots, \xi_{i_p}^\dagger)^\dagger [UU^\dagger](x_0, t) (\xi_{i_1}, \dots, \xi_{i_p})]. \quad (14)$$

It is obvious that in long time limit this sum will be dominated by the sum of the largest p eigenvalues of UU^\dagger . Introducing the cumulative Lyapunov exponents Λ_i as

$$\Lambda_i \equiv \lambda_1 + \dots + \lambda_i \quad (15)$$

one has

$$\Lambda_p = \lim_{t \rightarrow \infty} \frac{1}{2t} \langle \ln R_p(\mathbf{x}_0, t) \rangle. \quad (16)$$

Expression (16) is not suitable to be written as an integral of an exponential (a partition function) because of the logarithm. This is the usual quenched versus annealed problem in disordered systems: we are interested in the average of the logarithm and not the logarithm of the average. Below we shall overcome it by means of supersymmetry, but we shall first also give expressions for the annealed quantities.

The generalized Lyapunov exponents (see [5]) can be used to measure intermittency (i.e. rare trajectories having unusual Lyapunov exponents) and are relevant quantities in the thermodynamical formalism of chaotic systems. They are defined via the moments of the $R_p(\mathbf{x}_0, t)$

$$\Lambda_p^{2q} \equiv \lambda_1^{2q} + \dots + \lambda_p^{2q} = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \langle (R_p(\mathbf{x}_0, t)^q) \rangle. \quad (17)$$

To the extent that the moments $\langle (R_p(\mathbf{x}_0, t)^q) \rangle$ are sufficient to reconstruct the distribution law of $R_p(\mathbf{x}_0, t)$ and the averages for all real q , one can use them to find the quenched average (16). In such cases the quenched quantities can be extracted from (17) using the replica trick:

$$\Lambda_p = \frac{d\Lambda_p^q}{dq} (q = 0). \quad (18)$$

2.2. Generalized exponents

Let us first obtain an expression for the annealed exponents λ_i^{2q} for integer q . This calculation is the closest to the construction of Gozzi and Reuter [2], with the important difference that we consider the limit eigenvalues of the matrix $A = UU^\dagger$ and not of U itself.

Introducing p pairs of fermions a_i and b_i one can write (see appendix A)

$$R_p(\mathbf{x}_0, t) = \langle -|\psi_p \mathcal{T} \left[\exp \left(\int_0^t H_1(x(t')) dt' \right) \right] \psi_p^\dagger | - \rangle \quad (19)$$

where \mathcal{T} denotes time order and

$$H_1(t) \equiv V(\mathbf{x})_{ij}(a_i^\dagger a_j + b_i^\dagger b_j) \quad \psi_p \equiv \frac{1}{\sqrt{p!}} \sum_{i_1, \dots, i_p} a_{i_1} \cdots a_{i_p} b_{i_1} \cdots b_{i_p}. \quad (20)$$

The explicit time dependence of H_1 is given by the evolution of \mathbf{x} (2) via $V_{ij}(\mathbf{x})$. More generally, one can write, in terms of q replicas of the set of fermions a_i^γ and b_i^γ

$$R_p^q(\mathbf{x}_0, t) = \langle -|\psi_p^q \mathcal{T} \left[\exp \left(\int_0^t H_1^q(x(t')) dt' \right) \right] \psi_p^{q\dagger} | - \rangle \quad (21)$$

where

$$H_1^q(t) \equiv V(\mathbf{x})_{ij}(a_i^{\gamma\dagger} a_j^\gamma + b_i^{\gamma\dagger} b_j^\gamma) \quad \psi_p^q \equiv \frac{1}{\sqrt{p!}} \prod_{\gamma=1}^q \sum_{i_1, \dots, i_p} a_{i_1}^\gamma \cdots a_{i_p}^\gamma b_{i_1}^\gamma \cdots b_{i_p}^\gamma. \quad (22)$$

This is not the final expression, since we have not yet imposed that \mathbf{x} evolves according to the equation of motion.

The use of two sets (a and b) of fermions is needed in order to follow the rate of growth of the norm of vectors in the tangent space and not the evolution of the vectors themselves: we are interested in the eigenvalues of $U^\dagger U$ and not in those of U . This is obviously not a problem in dimension 1 [1]. In dimensions larger than 1, Gozzi and Reuter [2] introduced one family of fermions and thus studied the eigenvalues of U : these do not in general coincide with the Lyapunov exponents (see [14, 15] for a discussion), although they surely give relevant information. In section 5 we will explicitly show in an example how the a and b fermions interfere in a non-trivial way.

We now write an expression for the average over noise and/or initial conditions of (21) using the information that $\mathbf{x}(\mathbf{x}_0, \mathbf{t})$ evolves according to (2), and the probability density follows (4). One way to do this is to express the weight of each trajectory as

$$\begin{aligned} \langle R_p^q(\mathbf{x}_0, t) \rangle &= \left\langle \int P(\mathbf{x}_0) d\mathbf{x}_0 D\mathbf{x}' \delta(\mathbf{x}' - \mathbf{x}(\mathbf{x}_0, \eta, t')) \right. \\ &\quad \left. \times \langle -|\psi_p^q \left[\exp \left(\int_0^t H_1^q(x(t')) dt' \right) \right] \psi_p^{q\dagger} | - \rangle \right\rangle_\eta \end{aligned} \quad (23)$$

where Dx means the flat functional integral over trajectories, the delta-function imposes that $x(t)$ satisfies the equation of motion and $P(\mathbf{x}_0)$ is the initial condition distribution. This passage is just the standard textbook exercise of going from a Langevin to a Fokker–Planck description (see, for example, [16]). The result is that the probability evolves through the Fokker–Planck equation, and we have

$$\langle R_p^q(\mathbf{x}_0, t) \rangle = \langle 1 | \otimes \langle -|\psi_p^q [e^{-tH_q}] |\psi_p^{q\dagger} | - \rangle \otimes | P \rangle \quad (24)$$

where $H_q = H_{\text{FP}} - H_1^q$ is given in (7). The ket $\langle 1 |$ is the flat measure $\langle 1 | \mathbf{x} \rangle = 1$ and $\langle \mathbf{x} | P \rangle = P(\mathbf{x})$. Note that the time ordering is automatic, as the evolution of \mathbf{x} is taken care of by H_{FP} . In the limit $t \rightarrow \infty$ the logarithm of $\langle R_p^q(\mathbf{x}_0, t) \rangle$ is given by the smallest eigenvalue of H_q in the subspace having a non-zero overlap with the vectors $\langle 1 | \otimes \langle -|\psi_p^q$ and $\psi_p^{q\dagger} | - \rangle \otimes | P \rangle$.

We can now describe a practical algorithm for computing the generalized Lyapunov exponents. First we have to identify the smallest invariant subspace containing the states appearing to the right and to the left of (24). Clearly, the fermion numbers $N_a^\gamma = \sum_i a_i^{\gamma\dagger} a_i^\gamma$ and $N_b^\gamma = \sum_i b_i^{\gamma\dagger} b_i^\gamma$ are conserved, and the calculation of $\langle R_p^q(\mathbf{x}_0, t) \rangle$ involves working in the subspace $N_b^\gamma = N_a^\gamma = p$. Furthermore, because H_q and ψ_p^q commute with the operators \mathcal{P}^γ defined by

$$\mathcal{P}^\gamma a_i^\gamma \mathcal{P}^{\gamma\dagger} = -b_i^\gamma \quad \mathcal{P}^\gamma b_i^\gamma \mathcal{P}^{\gamma\dagger} = a_i^\gamma \quad (25)$$

we should look into the subspace of eigenfunctions symmetric under exchange of a and b , having eigenvalue one under the \mathcal{P}^ν : *this is where two families of fermions get mixed*. Because the quantity (24) is by construction positive, one can see that the eigenvalue having the smallest real part within this subspace has zero imaginary part.

2.2.1. Variational calculations. A variational approach can immediately be implemented for the calculation of the generalized (and, with an act of faith, the usual) Lyapunov exponents. Since one is looking for the lowest eigenvalue of H_q within a given subspace, one can use a variational trial function $|\varphi\rangle$ and make the variational estimation

$$\Lambda_i^{2q} \sim \min_{\varphi} \operatorname{Re} \frac{\langle \varphi | H_q | \varphi \rangle}{\langle \varphi | \varphi \rangle}. \quad (26)$$

If the family of variational functions is parametrized for every i, q , one obtains an approximation which is an explicit function of q : one can then envisage computing also an approximation for the usual Lyapunov exponents via (17).

2.3. Ordinary (quenched) Lyapunov exponents

In this section, we construct an expression for the (quenched) Lyapunov exponents using a supersymmetry formalism. For clarity, we shall do so in two steps: first a naive calculation that does not take into account the convergence of the sums, but is closer to the standard supersymmetry treatments in other contexts. Next, we use the Borel transform technique to work more properly.

2.3.1. Careless calculation. We introduce a set of fermion and boson operators as in (5) and (6), and define the number operators

$$\begin{aligned} N_a &= a_k^\dagger a_k & N_b &= b_k^\dagger b_k & N_\alpha &= \alpha_k^\dagger \alpha_k & N_\beta &= \beta_k^\dagger \beta_k \\ N_{\text{bos}} &= \frac{N_\alpha + N_\beta}{2} & N_{\text{fer}} &= \frac{N_a + N_b}{2} & \bar{N} &= N_{\text{fer}} + N_{\text{bos}} \end{aligned} \quad (27)$$

which will commute with all other operators. We shall also need

$$f \equiv a_k^\dagger b_k^\dagger \quad \bar{f} \equiv \alpha_k^\dagger \beta_k^\dagger. \quad (28)$$

Let us now introduce, for a given trajectory $\mathbf{x}(t)$, the quantity

$$H_1^S(t) = V_{kl}(\mathbf{x})(a_k^\dagger a_l + b_k^\dagger b_l + \alpha_k^\dagger \alpha_l + \beta_k^\dagger \beta_l). \quad (29)$$

With this notation we compute $Z(\mu, \bar{\mu})$ as

$$\langle - | e^{f^\dagger + \bar{f}^\dagger} \left(\mathcal{T} \exp \left(\int dt (H_1^S - \bar{\mu} N_{\text{bos}} - \mu N_{\text{fer}}) \right) (-1)^{N_{\text{bos}}} e^{f + \bar{f}} | - \rangle \quad (30)$$

and one can easily show that (appendix B)

$$Z(\mu, \bar{\mu}) = \frac{\det[1 + e^{-\mu t} A]}{\det[1 + e^{-\bar{\mu} t} A]} \quad (31)$$

where $A(t)$ is associated with the trajectory \mathbf{x} . This function will generate all the Lyapunov exponents for the trajectory as

$$G(\mu) \equiv - \lim_{t \rightarrow \infty} \frac{1}{t} \frac{\partial Z(\mu, \bar{\mu})}{\partial \mu} \Big|_{\bar{\mu}=\mu} = \lim_{t \rightarrow \infty} \sum_{j=1}^N \frac{e^{-\mu t} A_j}{1 + e^{-\mu t} A_j}. \quad (32)$$

Then, for large t , $G(\mu)$ is a ladder with steps when μ equals $2\lambda_i$ (i.e. it is the integral of the Lyapunov distribution function). Just as in the previous subsection, we wish to calculate $\langle G(\mu) \rangle$. Again, this is directly done in the Fokker–Planck formalism by the quantity

$$\langle G(\mu) \rangle \equiv - \lim_{t \rightarrow \infty} \frac{1}{t} \frac{\partial \langle Z(\mu, \bar{\mu}) \rangle}{\partial \mu} \Big|_{\bar{\mu}=\mu} \quad (33)$$

with

$$\langle Z(\mu, \bar{\mu}) \rangle = \langle \phi_L | \exp(-t(H_S + \mu N_{\text{fer}} + \bar{\mu} N_{\text{bos}})) (-1)^{N_{\text{bos}}} | \phi_R \rangle \quad (34)$$

where $H_S = H_{\text{FP}} - H_1^S$ is given in (6). The left and right eigenvectors are

$$|\phi_R\rangle = e^{f+\bar{f}} |-\rangle \otimes |P\rangle \quad \langle \phi_L| = \langle 1| \otimes \langle -| e^{f^\dagger + \bar{f}^\dagger}. \quad (35)$$

We can also write $\langle G(\mu) \rangle$ as an expectation value (see appendix C)

$$\langle G(\mu) \rangle = \langle \phi_L | e^{-t(H_S + \mu \bar{N})} N_{\text{fer}} (-1)^{N_{\text{bos}}} | \phi_R \rangle. \quad (36)$$

Equation (31) can be expressed as a formal series in powers of $\exp(-t\mu)$ and $\exp(-t\bar{\mu})$. This same series is reproduced by (30), or, in an averaged version, (34). Each term of the form $\exp[-t(n_1\mu + n_2\bar{\mu})]$ corresponds to an expectation value in the subspace of n_1 fermions and n_2 bosons. Clearly, (31) has a very small convergence radius: this is because the number of bosons, unlike the number of fermions, is unlimited. In other words, expressions (31), (30) and (34) have only a formal meaning. One can still work with them if at the end of a derivation one can resum the series exactly, in which case one has in fact performed the analytic continuation in μ and $\bar{\mu}$.

2.3.2. Borel transform. Let us now give a more proper construction of the quantity $G(\mu)$. It will turn out that the formalism that emerges is not much more complicated than that in the previous paragraphs. Briefly, the Borel transform technique consists in going from the formal series

$$h(y) \sim a_0 + a_1 y + a_2 y^2 + \dots \quad (37)$$

to its convergent transform

$$h^B(y) = a_0 + \frac{a_1}{1!} y + \frac{a_2}{2!} y^2 + \dots \quad (38)$$

which can then be inverted. In our case, we shall take

$$h(y) \sim a_0 + a_1 y + a_2 y^2 + \dots = \frac{\det[1 + y e^{-\mu t} A]}{\det[1 + y e^{-\bar{\mu} t} A]} \quad (39)$$

(cf (31)), and define as the Borel-transformed partition function

$$Z^B(\mu, \bar{\mu}) = h^B(y)|_{y=1} \quad (40)$$

that is, we are dividing by $(n_1 + n_2)!$ the term having n_1 fermions and n_2 bosons. Repeating the construction above, it is easy to see that

$$\langle Z^B(\mu, \bar{\mu}) \rangle = \langle \phi_L^B | \exp(-t(H_S + \mu N_{\text{fer}} + \bar{\mu} N_{\text{bos}})) (-1)^{N_{\text{bos}}} | \phi_R^B \rangle \quad (41)$$

which is just like (34) except that the left and right eigenvectors are

$$|\phi_R^B\rangle = g(f + \bar{f}) |-\rangle \otimes |P\rangle \quad \langle \phi_L^B| = \langle 1| \otimes \langle -| g(f^\dagger + \bar{f}^\dagger) \quad (42)$$

where the function $g(x)$ is

$$g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n! \sqrt{n!}}. \quad (43)$$

We can now define as before

$$\langle G^B(\mu) \rangle \equiv - \lim_{t \rightarrow \infty} \frac{1}{t} \frac{\partial \langle Z^B(\mu, \bar{\mu}) \rangle}{\partial \mu} \Big|_{\bar{\mu}=\mu} \quad (44)$$

which we can also write as an expectation value

$$\langle G^B(\mu) \rangle = \langle \phi_L^B | e^{-t(H_S + \mu \bar{N})} N_{\text{fer}} (-1)^{N_{\text{bos}}} | \phi_R^B \rangle. \quad (45)$$

Interestingly enough, we can retrieve the information *directly* (without the need to antitransform) from $G^B(\mu)$, since one can show (appendix C) that

$$G^B(\mu) = \sum_{j=1}^N \theta(2\lambda_j - \mu) \quad (46)$$

again gives the same ladder function.

2.3.3. Supersymmetry. The Hamiltonian H_S is invariant under all the (supersymmetric) transformations rotating simultaneously the $\alpha_i, \beta_i, a_i, b_i$ so as to leave the quadratic form $f + \bar{f}$ and the fermion–boson part of H_S invariant. Then, expectation values can be written in the standard form

$$\langle O \rangle = \text{Tr}[(-1)^{N_{\text{fer}}} \mathbf{C} O] \quad (47)$$

with \mathbf{C} supersymmetric,

$$\mathbf{C} \equiv \exp(-t[H_S + (\mu + i\pi)(N_{\text{fer}} + N_{\text{bos}})]) | \phi_R^B \rangle \langle \phi_L^B |. \quad (48)$$

Both the original and the Borel-transformed versions have the same symmetries.

One can show that supersymmetry is responsible for the fact that $Z(\mu, \mu)$ and $Z^B(\mu, \mu)$ are independent of μ : the constancy of the normalization is indeed the underlying reason why we can use the method to obtain quenched averages (see Balents and Fisher [10] for a detailed discussion of the supersymmetry group and representations).

2.4. Discussion

The calculation of generalized exponents λ_i^{2q} is done, as we have seen, by computing the lowest eigenvalue within a subspace of the Hilbert space. This is because the large t limit automatically projects onto the corresponding eigenstate. Approximate and numerical methods for the estimation of the ground state of Schrödinger-like operators abound in the literature, we have already mentioned the variational principle.

If one wishes to extend these results for the quenched exponents, an analytic continuation to $q \rightarrow 0$ is needed. This is easily done (although with a leap of faith) when an explicit expression for all even q is available. Such will be the case in a variational, a perturbative or a mean-field computation.

On the other hand, the supersymmetry method yields the Lyapunov exponents without the need of any continuation. However, there is a price to pay: expressions (36) and (45) involve a sum of terms within subspaces of any number of bosons. In the Borel-transformed version (45) this sum is convergent for all finite t . However, one can see that the largest term corresponds to a boson number of the order of $\exp(\lambda_1 t)$, and this number grows as we consider larger times. In other words, we can only perform this sum for finite t , and only then make $t \rightarrow \infty$. Again, this is no problem if an analytic expression is available (perhaps as a result of an approximation), but it does seem problematic to attack a problem numerically this way.

3. Symmetries: pairing rule for (quasi-)Hamiltonian systems

Damped Hamiltonian systems are a particular case of the dynamics (2), which can be written as

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} - \gamma p_i + \sqrt{\gamma} \eta_i \quad (49)$$

where γ measures the intensity of the coupling to the bath. The quasi-symplectic structure of Hamilton's equations has consequences.

One of the proprieties we can easily infer from our formulation is the pairing rule for the Lyapunov spectrum. This pairing rule was proved for the dynamical systems defined in (49) but without noise (named 'quasi-Hamiltonian' by Dressler [17]); in our formalism it is the result of a particle-hole symmetry. This is most easily seen for the annealed exponents. In the case

$$\mathcal{H} = \sum_i^N \frac{p_i^2}{2} + \mathcal{V}(\mathbf{q}) \quad (50)$$

the V_{ij} (9) read

$$V = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\frac{\partial^2 \mathcal{V}}{\partial q_i \partial q_j} & -\gamma \mathbf{1} \end{pmatrix} \quad (51)$$

(here and in what follows $\mathbf{0}$ and $\mathbf{1}$ are the null matrix and the identity matrix in the $N \times N$ space). Denoting $x_i = q_i$ for $(i = 1, \dots, N)$ and $x_i = p_i$ for $(i = N + 1, \dots, 2N)$ and introducing the fermions a_i^γ, b_i^γ we can compute the moments $\langle R_p(\mathbf{x}_0, t)^q \rangle$, $1 \leq p \leq 2N$, and obtain all the $2N$ annealed Lyapunov exponents.

We consider the transformations

$$\bar{a}_i^{\gamma\dagger} = \epsilon_{ij} a_j^\gamma \quad \bar{b}_i^{\gamma\dagger} = \epsilon_{ij} b_j^\gamma \quad (52)$$

where

$$\epsilon = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}. \quad (53)$$

We obtain a new set of fermions and we can construct new quantities as in (22), \bar{H}_q and $\bar{\psi}_p|\bar{0}\rangle$; they will be linked to the unbarred quantities by

$$\bar{H}_q = H_q - 2q\gamma(N - N_{\text{fer}}) \quad (54)$$

$$\bar{\psi}_p|\bar{0}\rangle = K_p \psi_{2N-p}|0\rangle \quad (55)$$

(where K_p is a constant not depending on time). Both barred and unbarred quantities can be used to compute the generalized Lyapunov exponents and we have the additional equality

$$\langle \bar{R}_p(\mathbf{x}_0, t)^q \rangle = \langle K_p^2 e^{(N-p)2q\gamma} R_{2N-p}(\mathbf{x}_0, t)^q \rangle. \quad (56)$$

This particle-hole symmetry, a consequence of the symplectic structure of the evolution operator, was noted by Gozzi and Reuter [2]. One then gets

$$\Lambda_p^{2q} = \Lambda_{2N-p}^{2q} + (N - p)2q\gamma \quad (57)$$

which in terms of the individual exponents becomes

$$\lambda_n^{2q} = -2q\gamma - \lambda_{2N-n+1}^{2q}. \quad (58)$$

Using (18) and (15) we can infer that the Lyapunov exponents are symmetric around $-\frac{\gamma}{2}$. Note that this result holds for arbitrary $\eta_i(t)$, and not only in the case in which it is a white noise.

4. Macroscopic systems

An active field of research is the information that Lyapunov exponents can provide in extensive systems in the thermodynamic limit. The natural object to study is the Lyapunov density function

$$\rho(\mu) \equiv \sum_{i=1}^N \delta(\lambda_i - \mu) \quad (59)$$

or, better, the cumulative version

$$C(\mu) \equiv \int_{\mu}^{\infty} \rho(\lambda) d\lambda. \quad (60)$$

These quantities, along with the generalized Lyapunov exponents (17), often appear in the context of the so-called thermodynamical formalism [12]; in particular $C(0)$ is the sum of positive exponents, related to the Kolmogorov–Sinai (KS) entropy through the Pesin (in)equality. Let us emphasize again that our statistical-mechanical formalism provides a tool to compute and study relevant quantities and not an alternative formulation for the thermodynamic formalism.

The existence of a thermodynamic limit for the Lyapunov densities (or for the greatest Lyapunov exponent) has been conjectured by several authors [18, 19] (see also [20]). A problem immediately arises: the Lyapunov exponents are themselves the result of the limit $t \rightarrow \infty$ and the question as to whether this and the thermodynamic limit commute is not obvious. Typical examples when they do not are when there are macroscopic motions that take times diverging with the size. We shall find a clear example in the Hamiltonian mean-field model below (section 6): the particles organize in an anisotropic object which can turn collectively like a rotor. In contact with a bath there is diffusion of the collective angle, but, since the moment of inertia scales with N , collective motion is absent if we consider $N \rightarrow \infty$ before $t \rightarrow \infty$. Sinai [21] has shown that in this last order of limits the densities are well defined in a system of confined particles with pairwise interactions: we shall see that this result is very natural.

In the previous sections, we have shown that $C(\mu)$ is the large time limit of $C_t(\mu)$ with

$$C_t(\mu) = G^B(2\mu, t) = \frac{1}{2t} \frac{\partial \ln Z^B(2\mu, 2\bar{\mu}, t)}{\partial \mu} \Big|_{\mu=\bar{\mu}} \quad (61)$$

(the logarithm in the rhs has no effect, since the normalization is one).

Consider this expression: Z^B is a partition function associated with the ‘quantum’ Hamiltonian H_S , where the time plays the role of an inverse temperature, μ that of a chemical potential and $C_t(\mu)$ that of the derivative of a free energy density with respect to the chemical potential (i.e. a particle number per unit volume). The Lyapunov density is hence a form of compressibility.

As mentioned in the introduction, if the original problem is on a lattice and has nearest neighbour interactions, the fermions and bosons are also lattice variables interacting with the nearest neighbour variables (through V_{ij}). We have then a ‘quantum’ lattice problem with short-range interactions. On the other hand, the system could be off-lattice, and the x_i be a set of d -dimensional vectors describing the position of the particles interacting via short-range pair forces $f_i(\mathbf{x}) = \sum_j f(x_i - x_j)$. The variables $(a_i, b_i, \alpha_i, \beta_i)$ play the role of ‘spin’ degrees of freedom carried by quantum particles, both the direct and the spin–spin interaction are also short range.

All in all, we are asking whether a quantum theory with short-range interactions has a good thermodynamic limit with a well-defined free energy density. There is only one

non-standard feature if we ask for the $t \rightarrow \infty$ taken before the thermodynamic limit: this is, as we have seen, like asking in a statistical-mechanical problem about the zero temperature limit taken before the thermodynamic limit—sometimes a tricky question.

The arguments on extensivity become more subtle if we wish to study the thermodynamic limit of the *largest* Lyapunov exponent: this is like asking in a particle system not what is the chemical potential needed to create a certain particle density, but rather to create *a single particle in the whole system*: clearly this is a question of order $O(1/N)$. We shall return to this point in section 6.

An interesting special case is the behaviour of the Lyapunov exponents close to zero in a system with soft modes. In the present context this concerns the properties of a statistical-mechanic problem around $\mu = 0$, i.e. free from external chemical potential. For example, from (36) we have

$$\langle \rho(\lambda = 0) \rangle = \langle \phi_L^B | e^{-tH_S} N_{\text{fer}} \frac{(N_{\text{fer}} + N_{\text{bos}})}{2} | \phi_R^B \rangle. \quad (62)$$

A very intriguing possibility that immediately comes to mind when working in the present framework is that of studying universality properties at critical points using renormalization-group ideas and techniques.

5. Random matrices

As explained in [15] the asymptotics of many systems, from neural networks in the thermodynamical limit [20] to localization problems in disordered conductors [8], are often modelled by products of random matrices.

In this section, we use our formalism to derive some results already obtained for some random matrix models. For brevity we shall only do this in the pure fermion (annealed) case, although the supersymmetric approach can also be applied.

The main lesson we shall obtain is that these systems become, by virtue of the disorder, interacting fermion problems. As such, they can be very well attacked by some of the many methods devised for such cases: Feynman diagrams of course, but also a resummation such as Hartree–Fock.

5.1. Weak disorder expansion

The first model where we can show the power of our approach is that proposed by Derrida *et al* [8]. They study the weak disorder expansion of the quenched Lyapunov exponents for a product of the form

$$P = \prod_t U_t \quad U_t = B_0 + \epsilon B \quad (63)$$

where B_0 is a fixed matrix, B is a random matrix, and ϵ is a small parameter. We shall study the case with B a multi-dimensional, Gaussian white noise, with zero mean (as the finite mean can be safely included in the constant matrix)

$$\langle B_{ij}(t) \rangle = 0 \quad \langle B_{ij}(t_1) B_{kl}(t_2) \rangle = \langle B_{ij} B_{kl} \rangle \delta(t_1 - t_2). \quad (64)$$

Following Derrida we study the case with the matrix B_0 having non-degenerate, well-separated eigenvalues $\epsilon_1 > \epsilon_2 > \dots > \epsilon_N$.

The first step in order to use our formalism is the set-up of a continuous-time variant of the problem

$$U(dt) = 1 + B_0 dt + \epsilon B dt. \quad (65)$$

Next, we exponentiate this expression. Due to the non-continuous character of the random term B the correct form of the exponential is

$$U(dt) = \exp\left(B_0 dt + \epsilon B dt - \frac{\epsilon^2}{2}\langle B^2 \rangle dt\right). \quad (66)$$

This means that the evolution of U will be given by (as in (9))

$$V_{ij}(t) = B_{0ij} + \epsilon B_{ij}(t) - \frac{\epsilon^2}{2}\langle B_{ik} B_{kj} \rangle. \quad (67)$$

We can now derive the path-integral form of R_p^q as

$$\frac{1}{p!} \langle - | \prod_{\gamma=1}^q a_{i_1}^\gamma \cdots a_{i_p}^\gamma b_{i_1}^\gamma \cdots b_{i_p}^\gamma | \mathcal{T} \exp\left(-\int dt H\right) | \prod_{\gamma=1}^q a_{i_1}^{\gamma\dagger} \cdots a_{i_p}^{\gamma\dagger} b_{i_1}^{\gamma\dagger} \cdots b_{i_p}^{\gamma\dagger} | - \rangle \quad (68)$$

(γ is the replica index going from 1 to q). We used as a Hamiltonian

$$H = V_{ij}(x)(a_i^{\gamma\dagger} a_j^\gamma + b_i^{\gamma\dagger} b_j^\gamma). \quad (69)$$

The average $\langle R_p^q \rangle$ will be expressed by integrating the Gaussian noise B_{ij} in (68) which will transform H into

$$H' = H_0 + \epsilon^2 H_I \quad (70)$$

$$H_0 = B_{0ii}(a_i^{\gamma\dagger} a_i^\gamma + b_i^{\gamma\dagger} b_i^\gamma) \quad (71)$$

$$H_I = \frac{\langle B_{mn} B_{ij} \rangle}{2} (a_i^{\gamma\dagger} a_m^{\gamma'\dagger} a_n^{\gamma'} a_j^\gamma + b_i^{\gamma\dagger} b_m^{\gamma'\dagger} b_n^{\gamma'} b_j^\gamma + a_i^{\gamma\dagger} b_m^{\gamma'\dagger} b_n^{\gamma'} a_j^\gamma). \quad (72)$$

We have used the fact that we can diagonalize B_0 and work on that base (which will not change the fermionic states). We have also arranged the creators and the annihilators of fermions in normal form. Expression (72) can be proved to be the correct one by going back to the Suzuki–Trotter product, performing the averages there, and then reconstructing the continuous version. We are now in the possession of a time-independent Hamiltonian. Note that after integration of the noise we get a non-trivial result only because of the presence of two types of fermions: without them we would have lost the higher moments of the noise.

Now we can carry on the program presented in section 2.2. The replica trick is easy to implement for perturbation expansions, since these yield an explicit dependence as a polynomial in q of each term of the expansion.

We want the p th cumulative Lyapunov exponent, so, as seen from (22) we work in the sector with p fermions of each type and q replicas. First, the ground state of H_0 is

$$\psi_0 = \prod_{\gamma} \prod_i^p a_i^\gamma b_i^\gamma \quad (73)$$

and the corresponding eigenvalue

$$2q \sum_i^p \varepsilon_i. \quad (74)$$

It is easy to see that ψ_0 has a finite overlap with the fermionic states in (22) and this will remain true for the perturbed ground states.

The first non-zero order in ϵ is in ϵ^2 ; in order to compute it we must use first-order perturbation theory to obtain

$$\langle \psi_0 | H_I | \psi_0^\dagger \rangle = -q \langle B_{ij} B_{ji} \rangle + \frac{3q^2}{2} \langle B_{ii} B_{jj} \rangle. \quad (75)$$

For the second-order perturbation theory (which will give the contribution in ϵ^2) we must identify the states connected by the perturbation (see appendix D). We can now retain the terms linear in q , and hence obtain the quenched average (the coefficients of the linear term (see (17))

$$\begin{aligned} \Lambda_p = & \sum_{i=1}^p \epsilon_i - \frac{\epsilon^2}{2} \sum_{i=1}^p \sum_{j=1}^p \langle B_{ij} B_{ji} \rangle + \epsilon^4 \sum_{i=1}^p \sum_{j=1}^p \sum_{l=1}^p \sum_{m>p} \frac{\langle B_{ij} B_{mi} \rangle \langle B_{jl} B_{lm} \rangle}{\epsilon_j - \epsilon_m} \\ & - \frac{\epsilon^4}{2} \sum_{i=1}^p \sum_{j=1}^p \sum_{m>p} \sum_{n>p} \frac{\langle B_{im} B_{jn} \rangle \langle B_{mi} B_{nj} \rangle}{\epsilon_j + \epsilon_i - \epsilon_m - \epsilon_n}. \end{aligned} \quad (76)$$

This expansion is the continuous-time, Gaussian white noise equivalent of the expression obtained in [8].

5.2. Random symplectic matrices

Another model that can be revisited is the random matrix model introduced by Parisi and Vulpiani [22] (see also [15] and [23]) in order to mimic some systems that show strong chaos. A continuous-time version of this model, in the case of only one spatial degree of freedom, is a linearized evolution of the form

$$V(t) = \begin{pmatrix} 0 & 1 \\ \eta(t) & 0 \end{pmatrix} \quad (77)$$

where η is a Gaussian noise with a mean r and a deviation σ .

Using this definition we can compute the average

$$\langle R_1^q(t) \rangle = \int P(\eta) d\eta \langle \psi^q | \mathcal{T} \left[\exp \left(\int_0^t H(\eta(t')) dt' \right) \right] | \psi_1^{q\dagger} \rangle \quad (78)$$

where

$$\psi_1^q = \prod_{\gamma=1}^q (a_q^\gamma b_q^\gamma + a_p^\gamma b_p^\gamma) \quad (79)$$

and

$$H = \sum_{\gamma=1}^q (a_q^{\gamma\dagger} a_p^\gamma + b_q^{\gamma\dagger} b_p^\gamma) + \eta(t) \sum_{\gamma=1}^q (a_p^{\gamma\dagger} a_q^\gamma + b_p^{\gamma\dagger} b_q^\gamma). \quad (80)$$

This Hamiltonian form of the exponential quantities associated with the Lyapunov exponents can be checked to be true using small-time developments in the Suzuki–Trotter formula, just as in the previous section. We can integrate the noise in (78) and obtain

$$\langle R(t)_1^q \rangle = \langle \psi_1^q | e^{tH'} | \psi_1^{q\dagger} \rangle \quad (81)$$

where the averaged Hamiltonian can again be checked in the discretized version

$$H' = \sum_{\gamma=1}^q (a_q^{\gamma\dagger} a_p^\gamma + b_q^{\gamma\dagger} b_p^\gamma) + r \sum_{\gamma=1}^q (a_p^{\gamma\dagger} a_q^\gamma + b_p^{\gamma\dagger} b_q^\gamma) + \frac{\sigma^2}{2} \left(\sum_{\gamma=1}^q (a_p^{\gamma\dagger} a_q^\gamma + b_p^{\gamma\dagger} b_q^\gamma) \right)^2. \quad (82)$$

In order to compute the generalized Lyapunov exponent λ_1^{2q} we must diagonalize this Hamiltonian on the 3^q -dimensional basis

$$a_p^{\gamma\dagger} b_p^{\gamma\dagger} \quad a_q^{\gamma\dagger} b_q^{\gamma\dagger} \quad \frac{1}{\sqrt{2}} (a_p^{\gamma\dagger} b_q^{\gamma\dagger} + a_q^{\gamma\dagger} b_p^{\gamma\dagger}) \quad (83)$$

which spans the maximum closed subspace containing the states (79).

Let us note here that the Hamiltonian (82) for the case $q = 1$, on the basis (83) is given by the matrix

$$\begin{pmatrix} 0 & 0 & \sqrt{2} \\ \sigma^2 & 0 & \sqrt{2}r \\ \sqrt{2}r & \sqrt{2} & 0 \end{pmatrix} \quad (84)$$

essentially the same used by Anteneodo and Vallejos [23, 24] to compute λ_1^2 .

We make a redefinition of the fermions as [25]

$$a_q \rightarrow t_1 a_q \quad a_q^\dagger \rightarrow \frac{a_q^\dagger}{t_1} \quad a_p \rightarrow t_2 a_p \quad a_p^\dagger \rightarrow \frac{a_p^\dagger}{t_2}. \quad (85)$$

Let us concentrate on the zero-mean noise ($r = 0$). With

$$\frac{t_1}{t_2} = (\sigma^2)^{\frac{1}{3}} \quad (86)$$

the Hamiltonian (82) becomes

$$H' \rightarrow (\sigma^2)^{\frac{1}{3}} \left[\sum_{\gamma=1}^q (a_q^{\gamma\dagger} a_p^\gamma + b_q^{\gamma\dagger} b_p^\gamma) + \frac{1}{2} \left(\sum_{\gamma=1}^q (a_p^{\gamma\dagger} a_q^\gamma + b_p^{\gamma\dagger} b_q^\gamma) \right)^2 \right]. \quad (87)$$

Now the σ dependence of the greatest Lyapunov exponent is clear: at any strength of the zero-mean noise $\lambda_1^{2q} = c(2q)\sigma^{\frac{2}{3}}$. This dependence will remain the same for the quenched Lyapunov exponent

$$\lambda_1 = c\sigma^{\frac{2}{3}} \quad c = \frac{1}{2} \left. \frac{dc(q)}{dq} \right|_{q=0} \quad (88)$$

as seen from numerical simulations (see figure 1).

We were not able to compute an analytical expression for $c(2q)$. For small q it is easy to diagonalize H : for $q = 1$ we obtain $\lambda_1^2 = (2\sigma^2)^{\frac{1}{3}}$ and going to $q = 2$ gives the possibility of obtaining a quadratic approximation and, with the replica trick, a better estimation of c (see figure 1).

Let us now turn to the case in which the noise has positive average. We can transform (82) using transformations (85) with

$$\frac{t_1}{t_2} = \sqrt{r}. \quad (89)$$

The Hamiltonian becomes $H' = \sqrt{r}H''$ with

$$H'' = \sum_{\gamma=1}^q (a_q^{\gamma\dagger} a_p^\gamma + b_q^{\gamma\dagger} b_p^\gamma) + \sum_{\gamma=1}^q (a_p^{\gamma\dagger} a_q^\gamma + b_p^{\gamma\dagger} b_q^\gamma) + s^2 \left(\sum_{\gamma=1}^q (a_p^{\gamma\dagger} a_q^\gamma + b_p^{\gamma\dagger} b_q^\gamma) \right)^2 \quad (90)$$

where $s^2 = \frac{\sigma^2}{r^{3/2}}$. There is a crossover (Lima and Rahibe [26]) in the Lyapunov exponent dependence between the limits of small and large s (see figure 1).

Finally, let us give an example of the role played by spatial structures. We consider a N -dimensional system with a noise $\eta(t)$ (77) which will now be a symmetric, random, $N \times N$ matrix with correlations depending on the distance between ‘sites’ i and j :

$$\langle \eta_{ij}(t) \rangle = 0 \quad \langle \eta_{ij}(t) \eta_{kl}(t') \rangle = \gamma^2 (|i - j|)(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \delta(t - t'). \quad (91)$$

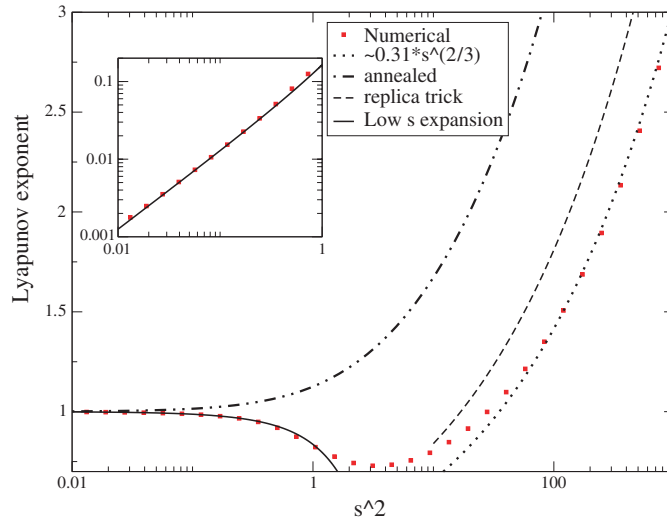


Figure 1. The greatest Lyapunov exponent versus the strength of the noise for a symplectic random matrix (the mean of the noise is 1). At $s \gg 1$ a replica approximation and a power law fit ($s^{2/3}$) are presented. The dot-dashed line is the annealed Lyapunov exponent ($\frac{\lambda_1^2}{2}$) while the dashed line is the approximation based on the first and the second generalized Lyapunov exponents. The straight line is the low s perturbative expansion (see appendix E); the inset shows the check of the expansion against the numerical data.

We shall compute the *annealed* Lyapunov exponent (λ_1^2). The Hamiltonian is

$$H = \sum_{l=1}^N (a_{q_l}^\dagger a_{p_l} + b_{q_l}^\dagger b_{p_l}) + \sum_{l,m=1}^N \eta_{lm} (a_{p_l}^\dagger a_{q_m} + b_{p_l}^\dagger b_{q_m}). \quad (92)$$

After averaging it becomes

$$H' = \sum_{l=1}^N (a_{q_l}^\dagger a_{p_l} + b_{q_l}^\dagger b_{p_l}) + \sum_{k \neq l=1}^N \frac{\gamma^2(|l-k|)}{2} (a_{p_l}^\dagger a_{q_k} + b_{p_l}^\dagger b_{q_k} + a_{p_k}^\dagger a_{q_l} + b_{p_k}^\dagger b_{q_l})^2 + \sum_{l=1}^N \gamma^2(0) (a_{p_l}^\dagger a_{q_l} + b_{p_l}^\dagger b_{q_l})^2. \quad (93)$$

The system becomes explicitly translational invariant. We have to diagonalize this Hamiltonian on the states

$$\frac{1}{\sqrt{N}} \sum_{l=1}^N a_{q_l}^\dagger b_{q_l}^\dagger, \quad \frac{1}{\sqrt{N}} \sum_{l=1}^N a_{p_l}^\dagger b_{p_l}^\dagger, \quad \frac{1}{\sqrt{2N}} \sum_{l=1}^N (a_{q_l}^\dagger b_{p_l}^\dagger + a_{p_l}^\dagger b_{q_l}^\dagger) \quad (94)$$

where N is the total number of sites. Within this subspace, H' is given by the matrix (84) with $r = 0$ and

$$\sigma^2 = N \left(\sum_{i=1}^N \gamma^2(i) + 2\gamma^2(0) \right) = N \sum_{j=1}^N \langle \eta_{ij} \eta_{ij} \rangle. \quad (95)$$

With this definition of σ , $\lambda_1^2 = (2\sigma^2)^{1/3}$ and this is the result obtained in [22].

6. Hamiltonian mean-field system

The Hamiltonian mean-field system we consider [27] is composed of N coupled rotators with a classical Hamiltonian

$$\mathcal{H} = \sum_i^N \frac{p_i^2}{2} + \frac{J}{N} \sum_{i,j} [1 - \cos(q_i - q_j)]. \quad (96)$$

The normalization of the coupling ensures an extensive free energy. This system has (in the canonical ensemble) a phase transition at $T_c = 0.5J$ between a ferromagnetic and a paramagnetic phase [27–29].

The equations of motion are, if we allow for noise and dissipation

$$\begin{aligned} \frac{dq_i}{dt} &= p_i \\ \frac{dp_i}{dt} &= -JM_x \sin q_i + JM_y \cos q_i - \gamma p_i + \sqrt{2\gamma T} \eta_i \end{aligned} \quad (97)$$

where we have introduced

$$M_x \equiv \frac{1}{N} \sum_i \cos q_i \quad M_y \equiv \frac{1}{N} \sum_i \sin q_i. \quad (98)$$

The matrix V_{ij} governing the evolution operator in tangent space is

$$V = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -E & -\gamma \mathbf{1} \end{pmatrix} \quad (99)$$

with

$$\begin{aligned} E_{ij} &= -\frac{J}{N} \cos(q_i - q_j) \quad \forall i \neq j \\ E_{ii} &= \frac{J}{N} \sum_i \cos(q_i - q_j) - \frac{J}{N} = JM_x \cos q_i + JM_y \sin q_i - \frac{J}{N}. \end{aligned} \quad (100)$$

We can now use the function defined in (30) in order to compute the Lyapunov spectrum. The supersymmetry Hamiltonian becomes a sum of single particle terms

$$H_S = \sum_i H_S^{(i)} \quad (101)$$

where the one particle Hamiltonian is

$$\begin{aligned} H_S^{(i)} &\equiv \frac{\partial}{\partial p_i} [JM_x \sin q_i - JM_y \cos q_i - \gamma p_i] \\ &+ T\gamma \frac{\partial^2}{\partial p_i^2} - p_i \frac{\partial}{\partial q_i} - Jv_{q_i}^{l\dagger} v_{p_i}^l - v_{p_i}^{l\dagger} v_{q_i}^l + \gamma v_{p_i}^{l\dagger} v_{p_i}^l \\ &+ J(M_x \cos q_i + M_y \sin q_i) v_{p_i}^{l\dagger} v_{q_i}^l - Jv_{p_i}^{l\dagger} \cos q_i C^l - Jv_{p_i}^{l\dagger} \sin q_i S^l \end{aligned} \quad (102)$$

(here no summation over i is implied). For compactness we have defined v^l for $l = 1, 2, 3, 4$ as

$$v_{q_i}^l \equiv (a_{q_i}, b_{q_i}, \alpha_{q_i}, \beta_{q_i}) \quad v_{p_i}^l \equiv (a_{p_i}, b_{p_i}, \alpha_{p_i}, \beta_{p_i}) \quad (103)$$

and the collective operators:

$$C^l \equiv \frac{1}{N} \sum_i v_{q_i}^l \cos q_i \quad S^l \equiv \frac{1}{N} \sum_i v_{q_i}^l \sin q_i. \quad (104)$$

Not surprisingly, the operator H_S is a mean-field (quantum) operator itself, and we can solve the problem with any of the usual methods. For example, introducing explicitly the collective variables using the functional delta-functions

$$\int D[M_x] \delta \left(NM_x - \sum_i \cos q_i \right) = \int D[M_x] D[\hat{M}_x] \exp \left(-\hat{M}_x \left(NM_x - \sum_i \cos q_i \right) \right) \quad (105)$$

$$\int D[M_y] \delta \left(NM_y - \sum_i \sin q_i \right) = \int D[M_y] D[\hat{M}_y] \exp \left(-\hat{M}_y \left(NM_y - \sum_i \sin q_i \right) \right) \quad (106)$$

$$\int D[C^l] \delta \left(NC^l - \sum_i v_{q_i}^l \cos q_i \right) = \int D[C^l] D[\hat{C}^l] \exp \left(-\hat{C}^l \left(NC^l - \sum_i v_{q_i}^l \cos q_i \right) \right) \quad (107)$$

$$\int D[S^l] \delta \left(NS^l - \sum_i v_{q_i}^l \sin q_i \right) = \int D[S^l] D[\hat{S}^l] \exp \left(-\hat{S}^l \left(NS^l - \sum_i v_{q_i}^l \sin q_i \right) \right) \quad (108)$$

we can write

$$\begin{aligned} \langle Z(\mu, \bar{\mu}) \rangle &= \int D[M_x] D[\hat{M}_x] \prod_l D[C^l] D[\hat{C}^l] D[S^l] D[\hat{S}^l] \\ &\quad \times \exp \left(-N \left(\int dt (\hat{M}_x M_x + \hat{M}_y M_y + \hat{C}^l C^l + \hat{S}^l S^l) - W \right) \right) \end{aligned} \quad (109)$$

where W is the action for a single pair of variables (q_i, p_i) and can be written as the average

$$e^W = \langle \phi_L^{(i)} | \exp(-t(H_{\text{eff}}^{(i)} + \mu N_{\text{fer}}^{(i)} + \bar{\mu} N_{\text{bos}}^{(i)})) (-1)^{N_{\text{bos}}^{(i)}} | \phi_R^{(i)} \rangle \quad (110)$$

(no summation on i).

The left and right eigenvectors (for one particle i) are

$$|\phi_R^{(i)}\rangle = \exp(f^{(i)\dagger} + \bar{f}^{(i)\dagger}) |-\rangle \otimes |P^{(i)}\rangle \quad \langle \phi_L^{(i)}| = \langle 1| \otimes \langle -| \exp(f^{(i)} + \bar{f}^{(i)}) \quad (111)$$

where for simplicity we have assumed that all pairs of variables (q_i, p_i) have the same initial distribution, and hence left and right vectors are in product form. From now on we drop the particle index i each time we describe only one particle. The single particle effective Hamiltonian ($H_{\text{eff}}^{(1)}$) equals

$$H_S^{(1)} - \hat{M}_x \cos q - \hat{M}_y \sin q - \hat{C}^l v_q^l \cos q - \hat{S}^l v_q^l \sin q \quad (112)$$

which has, in principle, a time dependence through the collective variables. We wish to evaluate $\langle Z(\mu, \mu') \rangle$ by the saddle-point method in the thermodynamical limit. The saddle-point equations for the ordinary collective variables read

$$\hat{M}_x = \frac{\partial W}{\partial M_x} \quad M_x = \frac{\partial W}{\partial \hat{M}_x} \quad \hat{M}_y = \frac{\partial W}{\partial M_y} \quad M_y = \frac{\partial W}{\partial \hat{M}_y}. \quad (113)$$

We also have saddle-point equations for the 16 fermionic and bosonic collective variables $\hat{C}^l, C^l, \hat{S}^l, S^l$.

We have 20 equations of type (113) which we can solve assuming that the system is at $t = 0$ already in thermodynamic equilibrium—the vector to the right in (110) is Gibbs-distributed. This implies that the collective variables may be constant in time, an assumption we verify later. First of all, it is easy to see that the saddle-point values of the \hat{C}^l , \hat{C}^l , \hat{S}^l and S^l vanish (appendix F). Next, one can see that if the endpoint of the trajectories is left free (i.e. we are not conditioning to a specific arrival point), then causality implies that $\hat{M}_x = \hat{M}_y = 0$ (appendix F). Under these assumptions, the only variables with non-zero saddle-point values are

$$M_x = \langle \cos q \rangle_M \quad M_y = \langle \sin q \rangle_M \quad (114)$$

where the average $\langle \bullet \rangle_M$ is taken with the single particle dynamics

$$\frac{dq}{dt} = p \quad \frac{dp}{dt} = -JM \sin q - \gamma p + \sqrt{2\gamma T} \eta. \quad (115)$$

Indeed, this yields the equilibrium value of M , as a solution of the equation

$$M = \frac{I_1(y)}{I_0(y)} \quad y = \frac{MJ}{T} \quad (116)$$

(where I are the Bessel functions, see [27]). What we have shown is that the Lyapunov density function of the system is, to leading order in N , the sum of Lyapunov densities for a single particle system moving according to (115). In the limit of zero coupling to the bath $\gamma = 0$ the exponents for a single particle are zero, as a consequence of conservation of energy and the pairing rule. In conclusion, in the thermodynamical limit the function $G(\mu)$ is a step of height $\sim 2N$ at zero.

Note that the fact that the energy of each particle is conserved separately is an artefact of the large N limit: there is a coupling between the fluctuations at the following order in $1/N$. On the other hand we have assumed that the equilibrium distribution is already established, even though this takes times that diverge with N .

A vanishing largest Lyapunov exponent λ_1 has already been obtained both numerically [28, 29] and analytically [24, 30] in the paramagnetic phase, but not in the ferromagnetic phase. This is not in contradiction with our results: according to our calculation one can still have a vanishing fraction of non-zero Lyapunov exponents, that do not contribute to the density function in the large N limit. If we wish to calculate the largest exponent λ_1 we should take our calculation to the next order.

Figure 2 shows a numerical calculation of the spectrum of Lyapunov exponents for various values of the number of particles N . The simulations are carried out in the microcanonical ensemble at an energy where the greatest Lyapunov exponent does not show important variation with N (the total energy $E = 0.3$, see [28]). Even though the computer time requirements do not allow us to get close to the thermodynamical limit, these numerical results show a reasonably good agreement with the hypothesis of a step spectrum.

At finite γ the Lyapunov distribution function is no longer a step, even in the thermodynamic limit. We can obtain it very easily by solving the one particle system (115) numerically. The results depend strongly on γ ; there is a scaling law in $\gamma^{1/3}$ below the critical temperature and an identical zero λ_1 in the paramagnetic phase (see figure 3).

Surprisingly, after rescaling the largest Lyapunov exponent for the one particle system (115) behaves very much like the Lyapunov exponent computed by Firpo [30] in the canonical ensemble. There are also qualitative and quantitative resemblances with the greatest Lyapunov exponent obtained in microcanonical simulations of a large (but obviously finite) number of particles (see [28, 29]). We can conclude that, at finite γ , $G(\mu)$ should have two sigmoids of height $\sim N$ located symmetrically around zero.

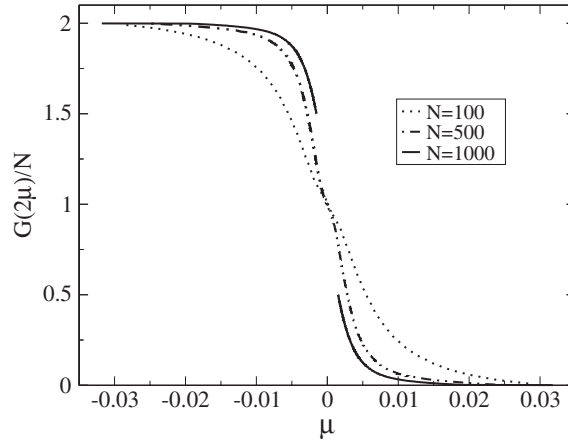


Figure 2. The microcanonical Lyapunov spectra of the Hamiltonian mean-field system with N particles at the total energy $E = 0.3$ presented as $\frac{G(2\mu)}{N}$ (see equation (61)) for different numbers of particles (N). Only the first halves of the spectra are computed, the other halves being obtained by the pairing rule.

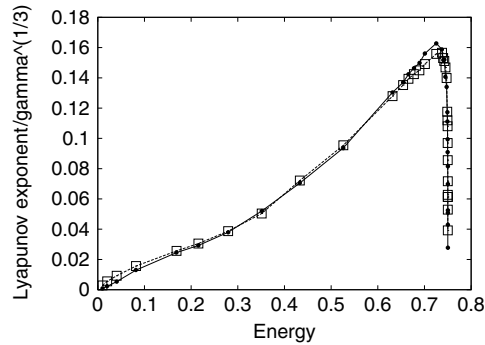


Figure 3. Lyapunov exponent for one particle with Kramers dynamics (115) versus the energy for $\gamma = 0.01$ (dots) and $\gamma = 0.001$ (squares) respectively. $E = 0.75$ is the energy at the transition point (for $J = 1$) and the Lyapunov exponent is automatically zero above this point. The exponents are scaled with $\gamma^{1/3}$.

7. Conclusions

Writing the Lyapunov exponents as a statistical-mechanics object provides a different perspective of the problem, allowing us to transpose much of the knowledge and intuition developed in that wider context.

Perhaps the most clear examples are the questions related to the thermodynamic limit. We have seen that the existence of a limit Lyapunov density function is the kind of extensivity property that most theoretical physicists would accept without proof—at least for the times that do not diverge with the system size.

As to the different approximation schemes, a case in question is the continuous product of random matrices studied in section 5. We have seen there that the problem is mapped into a system of interacting fermions. We have treated it as a perturbative expansion of weak interaction reobtaining the results presented in [8]. However, once in the language of

interacting particles, one immediately thinks of other, more global approximations; we have already mentioned Hartree–Fock, which can easily be implemented both in the pure fermion or in the supersymmetric formalism. Similarly, the standard mean-field treatment of the model of section 6 can be extended for the calculation of the exponents. This also suggests that it may be interesting in general to construct a local mean-field approximation for problems with space: this could give a simple analytic handle on the spatial structures involved with each exponent.

A question into which we have not looked in detail is intermittency. The generating function of generalized Lyapunov exponents λ_i^{2q} involves a distribution function in space that depends on q . Physically, this arises because we are conditioning the probability of a trajectory to have an unusual value of the exponents. Hence, studying the lowest eigenvector of H_q gives us information on the spatial structures responsible for intermittent behaviour.

Yet another interesting question is to look at systems at or near criticality, and borrow methods and ideas from the rich theory of critical phenomena to infer results on the behaviour of the exponents there.

Acknowledgments

We wish to thank Stefano Ruffo for clarification and encouragement at all stages of the preparation of this work and Frank Schmüser for a careful reading of the manuscript.

Appendix A

In this appendix we show how, using fermions, the quantity R defined in (14) can be written as a certain matrix element of the Hamiltonian (20) (equation (19)).

Let a_i^\dagger be a set of fermion operators

$$[a_i, a_j^\dagger]_+ = \delta_{ij} \quad (\text{A.1})$$

and we can encode (14) by writing

$$O(t) = a_k^\dagger y_k(t). \quad (\text{A.2})$$

We put

$$\dot{O}(t) = [V_{kl} a_k^\dagger a_l, O(t)]_- . \quad (\text{A.3})$$

It is easy to check that the y_i then satisfy (8). The solution to (A.3) is

$$O(t) = \left(\mathcal{T} \exp \left(\int dt V_{kl} a_k^\dagger a_l \right) \right) O(0) \left(\mathcal{T} \exp \left(\int dt V_{k'l'} a_{k'}^\dagger a_{l'} \right) \right)^{-1} \quad (\text{A.4})$$

which implies

$$\left(\mathcal{T} \exp \left(\int dt V_{kl} a_k^\dagger a_l \right) \right) a_i^\dagger \left(\mathcal{T} \exp \left(\int dt V_{k'l'} a_{k'}^\dagger a_{l'} \right) \right)^{-1} = U_{ji} a_j^\dagger. \quad (\text{A.5})$$

Let us now add a second set of fermions b_k^\dagger , and construct the operator

$$f \equiv a_k^\dagger b_k^\dagger \quad (\text{A.6})$$

which will evolve in time as

$$U_{ji} U_{ri} a_j^\dagger b_r^\dagger = \left(\mathcal{T} \exp \left(\int dt V_{kl} (a_k^\dagger a_l + b_k^\dagger b_l) \right) \right) f \left(\mathcal{T} \exp \left(\int dt V_{k'l'} (a_{k'}^\dagger a_{l'} + b_{k'}^\dagger b_{l'}) \right) \right)^{-1} \quad (\text{A.7})$$

(note that the same constructions can be made with bosons instead of fermions). Denoting $|-\rangle$ the fermion vacuum, (A.7) implies

$$\begin{aligned} & \langle -|a_{i_1} \cdots a_{i_p} b_{i_1} \cdots b_{i_p} \left| \mathcal{T} \left[\exp \left(\int dt V_{ij} (a_i^\dagger a_j + b_i^\dagger b_j) \right) \right] \right| a_{l_1}^\dagger \cdots a_{l_p}^\dagger b_{l_1}^\dagger \cdots b_{l_p}^\dagger |-\rangle \\ &= U_{k_1 l_1} \cdots U_{k_p l_p} U_{j_1 l_1} \cdots U_{j_p l_p} \langle -|a_{i_1} \cdots a_{i_p} a_{k_1}^\dagger \cdots a_{k_p}^\dagger |-\rangle \\ & \quad \times \langle -|b_{i_1} \cdots b_{i_p} b_{j_1}^\dagger \cdots b_{j_p}^\dagger |-\rangle \end{aligned} \quad (\text{A.8})$$

(this equality is true even if we do not sum over i_1, \dots, i_p and l_1, \dots, l_p).

Summing now over l_1, \dots, l_p we have

$$U_{k_1 l_1} U_{j_1 l_1} \cdots U_{k_p l_p} U_{j_p l_p} = (UU^\dagger)_{k_1 j_1} \cdots (UU^\dagger)_{k_p j_p}. \quad (\text{A.9})$$

Denoting with $\sigma((j_1, \dots, j_p) \rightarrow (k_1, \dots, k_p))$ the signature of the permutation between the two sets of indices when each set contains p different indices and the two sets are identical to a permutation, and zero otherwise, we can write

$$\langle -|a_{i_1} \cdots a_{i_p} a_{k_1}^\dagger \cdots a_{k_p}^\dagger |-\rangle = \sigma((i_1, \dots, i_p) \rightarrow (k_1, \dots, k_p)) \quad (\text{A.10})$$

which will give

$$\langle -|a_{i_1} \cdots a_{i_p} a_{k_1}^\dagger \cdots a_{k_p}^\dagger |-\rangle \langle -|b_{i_1} \cdots b_{i_p} b_{j_1}^\dagger \cdots b_{j_p}^\dagger |-\rangle = p! \sigma((j_1, \dots, j_p) \rightarrow (k_1, \dots, k_p)). \quad (\text{A.11})$$

Using (A.9) (A.8) and the properties of the signature it follows that (A.8) equals

$$(p!) \sum_{j_1, k_1, \dots, j_p, k_p} A_{j_1 k_1} \cdots A_{j_p k_p} \sigma((j_1, \dots, j_p) \rightarrow (k_1, \dots, k_p)) \quad (\text{A.12})$$

(remember that $A = UU^\dagger$). After summing on k_1, \dots, k_p expression (A.12) is, by the definition of the determinant,

$$(p!) \sum_{j_1, \dots, j_p} \det [(\xi_{j_1}^\dagger, \dots, \xi_{j_p}^\dagger)^\dagger [UU^\dagger] (\xi_{j_1}, \dots, \xi_{j_p})]. \quad (\text{A.13})$$

Appendix B

In this appendix, we derive formula (30). Denoting by $|-\rangle$ the fermion vacuum, (A.7) implies

$$\left(\mathcal{T} \exp \left(\int dt V_{kl} (a_k^\dagger a_l + b_k^\dagger b_l) \right) \right) e^f |-\rangle = e^{A_{jr} a_j^\dagger b_r^\dagger} |-\rangle \quad (\text{B.1})$$

and

$$\begin{aligned} & \langle -|e^{f^\dagger} \left(\mathcal{T} \exp \left(\int dt V_{kl} (a_k^\dagger a_l + b_k^\dagger b_l) - \mu N_{\text{fer}} t \right) \right) e^f |-\rangle = \langle -|e^{b_i a_i} e^{-t \mu N_{\text{fer}}} e^{A_{jr} a_j^\dagger b_r^\dagger} |-\rangle \\ &= \det[1 + e^{-\mu t} A]. \end{aligned} \quad (\text{B.2})$$

The last equality is most easily checked by performing a rotation of the fermions to diagonalize A . Then we have

$$\langle -|e^{b_i a_i} e^{-\mu N_{\text{fer}} t} e^{A_{jr} a_j^\dagger b_r^\dagger} |-\rangle = \langle -|e^{b_i^* a_i^*} e^{-\mu t} e^{A_j a_j^{\dagger*} b_j^{\dagger*}} |-\rangle = \prod_k \langle -|e^{b_k^* a_k^*} e^{-\mu t} e^{A_k a_k^{\dagger*} b_k^{\dagger*}} |-\rangle \quad (\text{B.3})$$

where \mathcal{A} are the eigenvalues of A (no summation on k); in the last expression there is no sum on i in the exponents. Each factor can be developed into $1 + e^{-\mu t} \mathcal{A}_i$, so the whole expression will be

$$\prod_i (1 + e^{-\mu t} \mathcal{A}_i) = \det[1 + e^{-\mu t} A]. \quad (\text{B.4})$$

Consider now the same steps, but now replacing the fermions a_k^\dagger and b_k^\dagger by bosons α_k^\dagger and β_k^\dagger as in (28); then we obtain the analogue of (B.2)

$$\begin{aligned} \langle -| e^{\bar{f}^\dagger} \left(\mathcal{T} \exp \left(\int dt V_{kl} (\alpha_k^\dagger \alpha_l + \beta_k^\dagger \beta_l) - \mu N_{\text{bos}} t \right) \right) e^{\bar{f}} | - \rangle &= \langle -| e^{\beta_i \alpha_i e^{-\mu t}} e^{A_j \alpha_j^\dagger \beta_j^\dagger} | - \rangle \\ &= \det[1 - e^{-\mu t} A]^{-1} \end{aligned} \quad (\text{B.5})$$

which can again be proved by performing a rotation of the bosons to diagonalize A . After the same step as in (B.3) the terms that survive are those with an equal number of the creation and destruction operators and we can use

$$\langle -| (\alpha^n \alpha^{\dagger n}) | - \rangle = n! \quad (\text{B.6})$$

to obtain the analogue of (B.4)

$$\prod_i (1 + e^{-\mu t} \mathcal{A}_i + e^{-2\mu t} \mathcal{A}_i^2 + \dots) = \prod_i \frac{1}{1 - e^{-\mu t} \mathcal{A}_i}. \quad (\text{B.7})$$

In order to change the sign in the denominator of the rhs of (B.7) we use the operator $(-1)^{N_{\text{bos}}}$ as

$$\frac{1}{\det[1 + A]} = \langle -| e^{\bar{f}^\dagger} \left(\mathcal{T} \exp \left(\int dt V_{kl} (\alpha_k^\dagger \alpha_l + \beta_k^\dagger \beta_l) \right) \right) (-1)^{N_{\text{bos}}} e^{\bar{f}} | - \rangle. \quad (\text{B.8})$$

Appendix C

In this appendix, we show the ladder structure of the functions G defined in (36) and (45). The common ingredient after developing the exponentials and matching the terms allowed by the conservation of bosons and fermions is

$$\begin{aligned} \langle -| (f^\dagger + \bar{f}^\dagger)^n \left(\mathcal{T} \exp \left(\int dt [V_{kl} (a_k^\dagger a_l + b_k^\dagger b_l + \alpha_k^\dagger \alpha_l + \beta_k^\dagger \beta_l) \right. \right. \\ \left. \left. - \mu (N_{\text{bos}} + N_{\text{fer}})] \right) \right) N_{\text{fer}} (-1)^{N_{\text{bos}}} (f + \bar{f})^n | - \rangle. \end{aligned} \quad (\text{C.1})$$

We can commute the ‘evolution’ operator with N_{fer} and apply it to the right ket; the action of the operator $N_{\text{fer}} + N_{\text{bos}}$ on the left ket can be easily computed so the expression becomes

$$e^{-n\mu t} \langle -| \frac{(f^\dagger - \bar{f}^\dagger)^n}{n!} N_{\text{fer}} \frac{(\mathcal{A}_i a_i^\dagger b_i^\dagger + \mathcal{A}_i \alpha_i^\dagger \beta_i^\dagger)^n}{n!} | - \rangle. \quad (\text{C.2})$$

The expansion of the powers in terms of individual creators and destructors is

$$e^{-n\mu t} (n!)^2 \sum_{k_1 + \dots + k_N = n} \langle -| \prod_j \frac{(b_j a_j - \beta_j \alpha_j)^{k_j}}{k_j!} N_{\text{fer}} \prod_j (\mathcal{A}_j)^{k_j} \frac{(a_j^\dagger b_j^\dagger + \alpha_j^\dagger \beta_j^\dagger)^{k_j}}{k_j!} | - \rangle. \quad (\text{C.3})$$

Here there is no sum on repeated indices inside the parentheses. Expanding further N_{fer} in terms of fermion operators this term can be written as

$$\begin{aligned} e^{-n\mu t} (n!)^2 \sum_l \sum_{\sum k_i = n} \langle -| \frac{(b_l a_l - \beta_l \alpha_l)^{k_l} (a_l^\dagger a_l + b_l^\dagger b_l) (a_l^\dagger b_l^\dagger + \alpha_l^\dagger \beta_l^\dagger)^{k_l}}{2k_l! k_l!} \mathcal{A}_l^{k_l} \prod_{j \neq l} \frac{(b_j a_j - \beta_j \alpha_j)^{k_j}}{k_j!} \\ \times \mathcal{A}_j^{k_j} \frac{(a_j^\dagger b_j^\dagger + \alpha_j^\dagger \beta_j^\dagger)^{k_j}}{k_j!} | - \rangle. \end{aligned} \quad (\text{C.4})$$

But for each l , at $j \neq l$ and $k_j > 0$ a factor of the product above will be

$$(-\mathcal{A}_j)^{k_j} \left(\frac{\langle -|\beta_j^{k_j} \alpha_j^{k_j} \alpha_j^{\dagger k_j} \beta_j^{\dagger k_j} |-\rangle}{k_j!^2} - \frac{k_j^2 \langle -|\beta_j^{k_j-1} \alpha_j^{k_j-1} \alpha_j^{\dagger k_j-1} \beta_j^{\dagger k_j-1} |-\rangle \langle -|b_j a_j a_j^\dagger b_j^\dagger |-\rangle}{k_j!^2} \right) \quad (\text{C.5})$$

and we can easily see that this equals zero. So, inside each term of the sum on l , only that with $k_l = n$ and the others $k_j = 0$ will survive.

The sum (C.3) simplifies to

$$\begin{aligned} e^{-n\mu t} (n!)^2 \sum_l \frac{\mathcal{A}_l^n (-1)^{n-1} n^2 \langle -|\beta_l^{n-1} \alpha_l^{n-1} \alpha_l^{\dagger n-1} \beta_l^{\dagger n-1} |-\rangle \langle -|b_l a_l \frac{a_l^\dagger a_l + b_l^\dagger b_l}{2} a_l^\dagger b_l^\dagger |-\rangle}{n!^2} \\ = e^{-n\mu t} \sum_l \mathcal{A}_l^n (-1)^{n-1}. \end{aligned} \quad (\text{C.6})$$

Now we can reconstruct the sum on n for the different cases; the Borel construction gives the series

$$-\sum_{j=1}^N \sum_{n=1}^{\infty} \frac{(-e^{-\mu t} \mathcal{A}_j)^n}{n!} = \sum_{j=1}^N [1 - \exp(-e^{-\mu t} \mathcal{A}_j)]. \quad (\text{C.7})$$

The series from (C.7) are convergent at each time and for each trajectory (in a well-behaved, smooth potential); one can formally sum the general terms (C.6) without the $\frac{1}{n!}$ and obtain

$$-\sum_{j=1}^N \sum_{n=1}^{\infty} (-e^{-\mu t} \mathcal{A}_j)^n = \sum_{j=1}^N \frac{e^{-\mu t} \mathcal{A}_j}{1 + e^{-\mu t} \mathcal{A}_j}. \quad (\text{C.8})$$

Using the exponential form of \mathcal{A}_j , in the limit $t \rightarrow \infty$, both (C.7) and (C.8) will give

$$G(\mu) = G^B(\mu) = \sum_{j=1}^N \theta(2\lambda_j - \mu). \quad (\text{C.9})$$

Appendix D

In this appendix, we derive the fourth term in ϵ of the weak disorder expansion treated in section 5.1. All the eigenstates of H_0 can be obtained by destroying and creating fermions in ψ_0 . Due to the fact that the perturbation contains two creators and two annihilators the states connected by the perturbation with ψ_0 are only those that differ from it by one or two fermions. The ‘replica composition’ of the states connected by the perturbation must be the same (two different replicas are not connected).

One set of states is formed by destroying one fermion from the first p and creating one (from the last $N - p$); both fermions must be in the same replica family

$$\psi_{b\gamma jm} = b_j^{\gamma\dagger} b_m^\gamma \psi_0 \quad \psi_{a\gamma jm} = a_j^{\gamma\dagger} a_m^\gamma \psi_0 \quad (\text{D.1})$$

where $j \leq p$, $m > p$ and $1 \leq \gamma \leq q$.

These eigenstates will contribute to the Lyapunov exponent by a second-order perturbation term

$$\frac{\langle \psi_{byjm} | H_I | \psi_0 \rangle \langle \psi_0 | H_I | \psi_{byjm} \rangle}{\varepsilon_j - \varepsilon_m} + \frac{\langle \psi_{ayjm} | H_I | \psi_0 \rangle \langle \psi_0 | H_I | \psi_{ayjm} \rangle}{\varepsilon_j - \varepsilon_m}. \quad (\text{D.2})$$

The linear part in q of this term is

$$2q \frac{\langle B_{ij} B_{mi} \rangle \langle B_{jl} B_{lm} \rangle}{\varepsilon_j - \varepsilon_m}. \quad (\text{D.3})$$

The states that have two different fermions are of three types

$$\begin{aligned} \psi_{byin;by'jm} &= b_i^{\gamma\dagger} b_j^{\gamma'\dagger} b_m^{\gamma'} b_n^{\gamma'} \psi_0 \\ \psi_{ayin;by'jm} &= a_i^{\gamma\dagger} b_j^{\gamma'\dagger} b_m^{\gamma'} a_n^{\gamma'} \psi_0 \\ \psi_{ayin;ay'jm} &= a_i^{\gamma\dagger} a_j^{\gamma'\dagger} a_m^{\gamma'} a_n^{\gamma'} \psi_0 \end{aligned} \quad (\text{D.4})$$

where $i \leq p$, $j \leq p$, $m > p$ and $n > p$.

Each of them will contribute with terms of the form

$$\frac{\langle \psi_{\dot{y}in;\dot{y}'jm} | H_I | \psi_{\dot{y}in;\dot{y}'jm} \rangle \langle \psi_0 | H_I | \psi_{\dot{y}in;\dot{y}'jm} \rangle}{\varepsilon_j + \varepsilon_i - \varepsilon_m - \varepsilon_n}. \quad (\text{D.5})$$

giving a linear term in q

$$q \frac{\langle B_{im} B_{jn} \rangle \langle B_{mi} B_{nj} \rangle}{\varepsilon_j + \varepsilon_i - \varepsilon_m - \varepsilon_n}. \quad (\text{D.6})$$

Appendix E

In this appendix, we apply the formula (76) to the perturbation expansion of the greatest Lyapunov exponent of the random symplectic matrix defined in (77):

$$V(t) = \begin{pmatrix} 0 & 1 \\ \eta(t) & 0 \end{pmatrix}. \quad (\text{E.1})$$

As seen from (90) it is enough to study the case of a Gaussian white noise $\eta(t)$ with mean 1 and deviation s . In this case, one can write

$$V(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + s \begin{pmatrix} 0 & 0 \\ \eta'(t) & 0 \end{pmatrix} \quad (\text{E.2})$$

where $\eta'(t)$ is now a Gaussian white noise of unit deviation and we will treat s as a small parameter. In the basis that diagonalizes the first matrix on the rhs of (E.2) we can write $V(t)$ as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + s\eta' \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad (\text{E.3})$$

We can now directly apply (76) with

$$B = \eta' \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad (\text{E.4})$$

and $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and obtain the perturbative expansion of the greatest Lyapunov exponent Λ_1

$$\Lambda_1 = 1 - \frac{s^2}{8} - \frac{5s^4}{128}. \quad (\text{E.5})$$

As we can see in figure 1 the expansion is very accurate up to $s = 1$.

Appendix F

In this appendix, we study the consistency equations for the collective variables in the Hamiltonian mean-field model. As we can see in equation (32) we need quantities at $\mu = \bar{\mu}$; the derivation implied by the definition of the G function will be carried out on $\langle Z(\mu, \bar{\mu}) \rangle$ after the saddle-point evaluation. So, we must take the derivative of expression (109). At this point, the derivative reads

$$\left. \frac{\partial Z^S(\mu, \bar{\mu})}{\partial \mu} \right|_{\bar{\mu}=\mu} = NZ^S \left. \frac{\partial W(\mu, \bar{\mu})}{\partial \mu} \right|_{\bar{\mu}=\mu} + \sum_k \left. \frac{\partial Z^S(\mu, \bar{\mu})}{\partial \mathcal{X}_k} \frac{\partial \mathcal{X}_k}{\partial \mu} \right|_{\bar{\mu}=\mu} \quad (\text{F.1})$$

where we take \mathcal{X}_k , for $k = 1, \dots, 20$, to be the vector of the collective variables, and Z^S is the value of $\langle Z(\mu, \bar{\mu}) \rangle$ at the saddle (109). We see that the second part of the rhs of this equation is zero because of the consistency equations (113); in conclusion, we are interested in the values of the collective variables at $\mu = \bar{\mu}$ and only the direct dependence on the two variables of W will be interesting for us. With this assumption we write the consistency equation for the boson and fermion variables as

$$S^l = \frac{\langle \phi_L | \exp(-t(H_{\text{eff}}^{(1)} + \mu \bar{N})) v_q^l \sin q (-1)^{N_{\text{bos}}} | \phi_R \rangle}{e^W} \quad (\text{F.2})$$

$$\hat{S}^l = \frac{\langle \phi_L | \exp(-t(H_{\text{eff}}^{(1)} + \mu \bar{N})) v_p^{l\dagger} \sin q (-1)^{N_{\text{bos}}} | \phi_R \rangle}{e^W} \quad (\text{F.3})$$

$$C^l = \frac{\langle \phi_L | \exp(-t(H_{\text{eff}}^{(1)} + \mu \bar{N})) v_q^l \cos q (-1)^{N_{\text{bos}}} | \phi_R \rangle}{e^W} \quad (\text{F.4})$$

$$\hat{C}^l = \frac{\langle \phi_L | \exp(-t(H_{\text{eff}}^{(1)} + \mu \bar{N})) v_p^{l\dagger} \cos q (-1)^{N_{\text{bos}}} | \phi_R \rangle}{e^W}. \quad (\text{F.5})$$

All those equations contain expectation values of *single* boson and fermion operators which necessarily vanish. This is normal, as we could expect from the beginning that those variables integrate out, contributing only with a non-exponential prefactor. This prefactor (as expected from the supersymmetry considerations) will be one in the limit $\mu = \bar{\mu}$ but in the process of derivation will give terms of order $O(N^{-1})$.

There are four equations left which can be divided into two parts. The first part contains the variables \hat{M}_x and \hat{M}_y

$$\hat{M}_x = J \frac{\langle \phi_L | e^{-t(H_{\text{eff}}^{(1)} + \mu \bar{N})} \left(\frac{\partial}{\partial p} \sin q + \cos q v_p^{l\dagger} v_q^l \right) (-1)^{N_{\text{bos}}} | \phi_R \rangle}{e^W} \quad (\text{F.6})$$

$$\hat{M}_y = -J \frac{\langle \phi_L | e^{-t(H_{\text{eff}}^{(1)} + \mu \bar{N})} \left(\frac{\partial}{\partial p} \cos q - \sin q v_p^{l\dagger} v_q^l \right) (-1)^{N_{\text{bos}}} | \phi_R \rangle}{e^W}. \quad (\text{F.7})$$

The first term on the rhs of these equations is zero (as demanded by the causality) and the second is also zero, due to the supersymmetry.

Finally, the last two equations are the only non-trivial ones

$$M_x = \frac{\langle \phi_L | e^{-t(H_{\text{eff}}^{(1)} + \mu \bar{N})} \cos q (-1)^{N_{\text{bos}}} | \phi_R \rangle}{e^W} \quad (\text{F.8})$$

$$M_y = \frac{\langle \phi_L | e^{-t(H_{\text{eff}}^{(1)} + \mu \bar{N})} \sin q (-1)^{N_{\text{bos}}} | \phi_R \rangle}{e^W}. \quad (\text{F.9})$$

Now the Hamiltonian $H_{\text{eff}}^{(1)}$ is much simpler and reads

$$H_{\text{eff}}^{(1)} = \frac{\partial}{\partial p}(JM \sin q - \gamma p) + T\gamma \frac{\partial^2}{\partial p^2} - p \frac{\partial}{\partial q} - Jv_q^{l\dagger} v_p^l + v_q^{l\dagger} v_p^l + \gamma v_p^{l\dagger} v_p^l + JM \cos q v_p^{l\dagger} v_q^l \quad (\text{F.10})$$

where we also used the rotational symmetry (which defines a saddle manifold) on the space of M_x and M_y which allows us to fix $M_y = 0$ and $M_x = M$; looking back at (109) and (F.1) we can conclude that

$$G(\mu) = N \lim_{t \rightarrow \infty} \frac{1}{t} e^{NW} \left. \frac{\partial W(\mu, \bar{\mu})}{\partial \mu} \right|_{\bar{\mu}=\mu}. \quad (\text{F.11})$$

Formula (110) combined with the simple expression of H_{eff} (F.10) allows us to infer that

$$G(\mu) = NG^{(1)}(\mu) \quad (\text{F.12})$$

where $G^{(1)}(\mu)$ characterize the Lyapunov spectrum of a single particle with the dynamics (115); this spectrum contains in fact only two exponents.

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